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Elastic Thermal Stresses in Delta Wings

Part II

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Contents

List of symbols

Symbols listed in Part I of this report are not repeated here.

b......... half base-length of wing

l......... wing chord

l_0......... distance between base and x-axis.

s .... exponent in representation of wing cross-section.

$\bar{\beta}$ .... angle at base of triangle. $\bar{\beta} = \frac{\pi}{2} - \alpha$.

$\mu(\zeta)$...... complex function replacing $\varphi(\zeta)$ of Part I

General Remarks

1.) Introduction. In Part I of this report[1] the basic equations for shallow conical shells have been derived assuming the temperature distribution in the shell to be symmetric with respect to the vertical plane of symmetry of the delta wing. Boundary conditions have been formulated and a perturbation method has been developed consisting of two steps termed first and second approximation, respectively. Temperature distribution has been resolved into two parts: one symmetric with respect to the horizontal plane of symmetry of the delta wing, and the other skew-symmetric with respect to that plane. The general solution and a numerical example have been given for the symmetric part using first approximation only.

In the present Part II of the report the study of thermal stresses in delta wings is continued. Equations of first approximation for the skew-symmetric temperature part are derived and solved. A numerical example is given. Then the equations of second (and final) approximation are obtained and solved for both the symmetric and the skew-symmetric part. The numerical example is continued into the second approximation.

Frequent reference is made in the following to Part I. Equations or figures contained in Part I are quoted by adding a Roman I to the equation number or figure number: Eq.(I-12-1) or Fig.I-3.

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1) Parkus (see list of references).

Skew-Symmetric Temperature Distribution

2.) General Equations. The two basic equations (I-7-2) remain, of course, valid in the present case of a temperature distribution skew-symmetric with respect to the x,y-plane, Fig.1. Boundary conditions (I-8-3), however, have to be changed. Since upper and lower shell now experience the same deflection w but opposite tangential displacements u and v, Fig.2, the conditions

$$\left.\begin{array}{ll} m_n = 0 & u = 0 \\ \bar{q}_n \equiv q_n + \dfrac{\partial m_{ns}}{\partial s} = \pm \dfrac{\partial W}{\partial n} n_n & v = 0 \end{array}\right\} \tag{2-1}$$

have to be satisfied along base (upper sign) and legs (lower sign) of the triangle, cf.Fig.1. Boundary conditions (I-8-4) along the y-axis (line of symmetry) remain unchanged:

$$\left.\begin{array}{ll} \dfrac{\partial w}{\partial x} = 0 & u = 0 \\ q_x = - \dfrac{\partial W}{\partial n} n_x & n_{xy} = 0 \end{array}\right\} \tag{2-2}$$

Equations (2-1) have now to be expressed in terms of the deflection w and the stress function F. Eqs.(I-7-5) and (I-7-6) render

$$\left.\begin{array}{l} m_\varphi = - N\left[\nabla^2 w - (1-\nu) \dfrac{\partial^2 w}{\partial r^2} + \tau_1\right] \\ m_{r\varphi} = - (1-\nu) N \dfrac{\partial}{\partial r}\left(\dfrac{1}{r} \dfrac{\partial w}{\partial \varphi}\right) \end{array}\right\} \tag{2-3}$$

Furthermore[1)]

$$q_\varphi = - \frac{N}{r} \frac{\partial}{\partial\varphi}(\nabla^2 w + \tau_1) \tag{2-4}$$

Base and leg of the triangle are given by $\varphi = 0$ and $\varphi = \bar{\beta} = \frac{\pi}{2} - \alpha$, respectively. Hence, the first two of conditions (2-1) take on the form

$$\left.\begin{aligned} &\nabla^2 w - (1-\nu)\frac{\partial^2 w}{\partial r^2} + \tau_1 = 0 \\ &\frac{\partial}{\partial\varphi}(\nabla^2 w + \tau_1) + (1-\nu) r \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial\varphi}\right) - \frac{1}{N}\frac{\partial w}{\partial\varphi}\frac{\partial^2 F}{\partial r^2} = 0 \end{aligned}\right\} \tag{2-5a}$$

along $\varphi = 0$ and $\varphi = \bar{\beta}$.

From eqs.(I-5-3) and (I-3-7) one finds using polar coordinates and denoting the radial displacement by u and the change of the angle φ by ε

$$\left.\begin{aligned} &\frac{\partial u}{\partial r} = \frac{1}{Eh}\left[\nabla^2 F - (1+\nu)\frac{\partial^2 F}{\partial r^2} + (1-\nu) D\,\tau_o\right] \\ &\frac{u}{r} + \frac{\partial\varepsilon}{\partial\varphi} + \frac{c}{r} w = \frac{1}{Eh}\left[(1+\nu)\frac{\partial^2 F}{\partial r^2} - \nu\nabla^2 F + (1-\nu) D\,\tau_o\right] \\ &\frac{\partial u}{\partial\varphi} + r^2\frac{\partial\varepsilon}{\partial r} = \frac{2(1+\nu)}{Eh}\frac{\partial}{\partial\varphi}\left(\frac{F}{r} - \frac{\partial F}{\partial r}\right) \end{aligned}\right\} \tag{2-6}$$

Along $\varphi = 0$ and $\varphi = \bar{\beta}$ we have $u = 0$ and $\varepsilon = 0$ for all values of r. Hence, from the first equation

$$\nabla^2 F - (1+\nu)\frac{\partial^2 F}{\partial r^2} + (1-\nu) D\,\tau_o = 0 \tag{2-5b}$$

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1) Melan-Parkus, p.59

Differentiating the first equation with respect to φ and the third with respect to r, and subtracting, there follows

$$\frac{\partial}{\partial\varphi}\left[\nabla^2 F + (1+\nu) r \frac{\partial^2}{\partial r^2}\left(\frac{F}{r}\right) + (1-\nu) D \tau_0\right] = 0 \qquad (2\text{-}5c)$$

along $\varphi = 0$ and $\varphi = \bar{\beta}$.

The four equations (2-5a) to (2-5c) replace the boundary conditions (2-1).

3.) Perturbation procedure. Expanding w and F in series in powers of c, Eqs.(I-9-2), and retaining only the first two terms one obtains eqs.(I-9-3) and (I-9-4):

$$\left.\begin{aligned} \nabla^2\nabla^2 w_1 &= \frac{p}{N} - \nabla^2 \tau_1 \\ \nabla^2\nabla^2 F_1 &= -(1-\nu) D \nabla^2 \tau_0 \end{aligned}\right\} \qquad (3\text{-}1)$$

$$\left.\begin{aligned} N\nabla^2\nabla^2 w_2 &= -\frac{1}{r}\frac{\partial^2 F_1}{\partial r^2} \\ \nabla^2\nabla^2 F_2 &= \frac{Eh}{r}\frac{\partial^2 w_1}{\partial r^2} \end{aligned}\right\} \qquad (3\text{-}2)$$

Performing the same expansion in the boundary conditions (2-5a) to (2-5c) and using (I-9-1) one finds

$$\left.\begin{aligned} \nabla^2 w_1 - (1-\nu)\frac{\partial^2 w_1}{\partial r^2} + \tau_1 &= 0 \\ \frac{\partial}{\partial\varphi}\left[\nabla^2 w_1 + (1-\nu) r \frac{\partial^2}{\partial r^2}\left(\frac{w_1}{r}\right) + \tau_1\right] &= 0 \end{aligned}\right\} \qquad (3\text{-}3)$$

$$\left.\begin{aligned} &\nabla^2 F_1 - (1+\nu)\frac{\partial^2 F_1}{\partial r^2} + (1-\nu) D\,\tau_0 = 0 \\ &\frac{\partial}{\partial\varphi}\left[\nabla^2 F_1 + (1+\nu) r \frac{\partial^2}{\partial r^2}\left(\frac{F_1}{r}\right) + (1-\nu) D\,\tau_0\right] = 0 \end{aligned}\right\} \qquad (3\text{-}4)$$

$$\left.\begin{aligned} &\nabla^2 w_2 - (1-\nu)\frac{\partial^2 w_2}{\partial r^2} = 0 \\ &\frac{\partial}{\partial\varphi}\left[\nabla^2 w_2 + (1-\nu) r \frac{\partial^2}{\partial r^2}\left(\frac{w_2}{r}\right)\right] = \frac{1}{N}\frac{\partial f}{\partial\varphi}\frac{\partial^2 F_1}{\partial r^2} \end{aligned}\right\} \qquad (3\text{-}5)$$

$$\left.\begin{aligned} &\nabla^2 F_2 - (1+\nu)\frac{\partial^2 F_2}{\partial r^2} = 0 \\ &\frac{\partial}{\partial\varphi}\left[\nabla^2 F_2 + (1+\nu) r \frac{\partial^2}{\partial r^2}\left(\frac{F_2}{r}\right)\right] = 0 \end{aligned}\right\} \qquad (3\text{-}6)$$

The four pairs of boundary conditions (3-3) to (3-6) are valid along base and legs of the triangle, i.e. for $\varphi = 0$ and $\varphi = \bar{\beta}$. The remaining four conditions (2-2), valid along the line of symmetry $x = 0$, take on the form (I-9-7) and (I-9-8) and, as in the case of the symmetric temperature distribution, can be taken care of automatically by extending the validity of the preceding equations to the entire triangular region of the wing. For the second approximation, however, an additional external load

$$q = -\,2l\,\frac{\partial f}{\partial x}\,n_{x1} \qquad (3\text{-}7)$$

has to be placed along the y-axis, cf.eq.(I-9-9).

<u>4.) First approximation.</u> The solutions of eqs.(3-1) are split up into particular solutions w_1^p, F_1^p of the complete equations and solutions w_1^h, F_1^h of the homogeneous equations. If, as in Part I of this report, attention is restricted to temperature changes only then $p = 0$ in eq.(3-1) and the particular solutions may be obtained from eqs.(I-10-2):

$$\left.\begin{aligned} \nabla^2 w_1^p &= -\tau_1 \\ \nabla^2 F_1^p &= -(1-\nu) D\, \tau_o \end{aligned}\right\} \qquad (4\text{-}1)$$

No boundary conditions need be taken into consideration.

Eqs.(4-1) can be solved by the same method as used in sec.10 of Part I. Another possibility consists in employing the method of finite differences.

The particular solutions thus obtained will, in general, not satisfy the boundary conditions (3-3) and (3-4). Substituting $w_1 = w_1^p + w_1^h$ into (3-3) and making use of eqs.(4-1) one obtains

$$\left.\begin{aligned} \nabla^2 w_1^h - (1-\nu)\frac{\partial^2 w_1^h}{\partial r^2} &= (1-\nu)\frac{\partial^2 w_1^p}{\partial r^2} \\ \frac{\partial}{\partial \varphi}\left[\frac{1}{r}\nabla^2 w_1^h + (1-\nu)\frac{\partial^2}{\partial r^2}\left(\frac{w_1^h}{r}\right)\right] &= -(1-\nu)\frac{\partial^3}{\partial\varphi\partial r^2}\left(\frac{w_1^p}{r}\right) \end{aligned}\right\} \qquad (4\text{-}2)$$

The same functional representation is assumed for w_1^h as given by eq.(I-11-3)

$$2w_1^h = \bar{z}\mu(z) + \overline{z\mu(z)} + \chi(z) + \overline{\chi(z)} \qquad (4\text{-}3)$$

where $\mu(z)$ and $\chi(z)$ are analytic functions[1] of the complex variable

$$z = x + iy = re^{i\varphi} - a, \quad a = l \tan\alpha + il_o \tag{4-4}$$

A bar denotes the conjugate complex quantity. Using

$$\frac{\partial}{\partial r} = e^{i\varphi}\frac{\partial}{\partial z} + e^{-i\varphi}\frac{\partial}{\partial \bar{z}}, \quad \frac{1}{r}\frac{\partial}{\partial \varphi} = i(e^{i\varphi}\frac{\partial}{\partial z} - e^{-i\varphi}\frac{\partial}{\partial \bar{z}})$$

and writing

$$\chi'(z) = \psi(z) \tag{4-5}$$

one finds

$$\nabla^2 w_1^h = 2\left[\mu'(z) + \overline{\mu'(z)}\right] \tag{4-6}$$

$$\left.\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial\varphi}\nabla^2 w_1^h &= 2i\left[\mu''(z)e^{i\varphi} - \overline{\mu''(z)}e^{-i\varphi}\right]\\
\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w_1^h}{\partial\varphi}\right) &= \frac{i}{2}\left[\bar{z}\mu''(z) + \psi'(z)\right]e^{2i\varphi} -\\
&\quad - \frac{i}{2}\left[z\overline{\mu''(z)} + \overline{\psi'(z)}\right]e^{-2i\varphi}
\end{aligned}\right\} \tag{4-7}$$

$$\frac{\partial^2 w_1^h}{\partial r^2} = \frac{1}{2}\left[\bar{z}\mu''(z) + \psi'(z)\right]e^{2i\varphi} + \frac{1}{2}\left[z\overline{\mu''(z)} + \overline{\psi'(z)}\right]e^{-2i\varphi} + \\ + \mu'(z) + \overline{\mu'(z)} \tag{4-8}$$

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1) The letter μ is now used instead of φ to avoid confusion with the polar angle.

In order to simplify the boundary conditions the second of eqs.(4-2) is multiplied by $dr = e^{-i\varphi}dz = e^{i\varphi}d\bar{z}$, and integrated:

$$2i\left[\mu'(z) - \overline{\mu'(z)}\right] + \frac{i}{2}(1-\nu)\left\{\left[\bar{z}\mu''(z) + \psi'(z)\right]e^{2i\varphi} - \right.$$

$$\left. - \left[\overline{z\mu''(z)} + \overline{\psi'(z)}\right]e^{-2i\varphi}\right\} = -(1-\nu)\left[\frac{\partial^2}{\partial\varphi\partial r}\left(\frac{w_1^p}{r}\right) + C_1(\varphi)\right]$$

C_1 is real. The first of eqs.(4-2) assumes the form

$$(1+\nu)\left[\mu'(z) + \overline{\mu'(z)}\right] - \frac{1-\nu}{2}\left\{\left[\bar{z}\mu''(z) + \psi'(z)\right]e^{2i\varphi} + \right.$$

$$\left. + \left[\overline{z\mu''(z)} + \overline{\psi'(z)}\right]e^{-2i\varphi}\right\} = (1-\nu)\frac{\partial^2 w_1^p}{\partial r^2}$$

Multiplying this equation by i and adding it to equation above renders

$$(3+\nu)\mu'(z) - (1-\nu)\overline{\mu'(z)} - (1-\nu)\left[\overline{z\mu''(z)} + \overline{\psi'(z)}\right]e^{-2i\varphi}$$

$$= (1-\nu)\left[\frac{\partial^2 w_1^p}{\partial r^2} + i\frac{\partial^2}{\partial r\partial\varphi}\left(\frac{w_1^p}{r}\right) + i\,C_1(\varphi)\right]$$

This equation is now once more integrated with respect to r. One has with eq.(4-4)

$$\int\overline{z\mu''(z)}e^{-2i\varphi}dr = \int(\bar{z} + \bar{a} - ae^{-2i\varphi})\overline{\mu''(z)}dr =$$

$$= e^{i\varphi}\int\overline{z\mu''(z)}d\bar{z} + (\bar{a}e^{i\varphi} - ae^{-i\varphi})\int\overline{\mu''(z)}d\bar{z}$$

$$= e^{i\varphi}\left[\overline{z\mu'(z)} - \overline{\mu(z)}\right] + (\bar{a}e^{i\varphi} - ae^{-i\varphi})\overline{\mu'(z)} = e^{-i\varphi}\overline{z\mu'(z)} - e^{i\varphi}\overline{\mu(z)}$$

and hence

$$\lambda\mu(z) - z\overline{\mu'(z)} - \overline{\psi(z)} = e^{i\varphi}\left[\frac{\partial w_1^p}{\partial r} + i\,\frac{\partial}{\partial\varphi}\left(\frac{w_1^p}{r}\right)\right] + i\,C_1(\varphi)z + C_2(\varphi)$$

where

$$\lambda = \frac{3+\nu}{1-\nu} \tag{4-9}$$

The triangular region is simply connected and C_1 and C_2 may be put equal to zero. Therefore the boundary condition reads finally

$$\lambda\mu(z) - \overline{z\mu'(z)} - \overline{\psi(z)} = \left(\frac{\partial w_1^p}{\partial r} + i\,\frac{1}{r}\,\frac{\partial w_1^p}{\partial\varphi}\right)e^{i\varphi} \tag{4-10}$$

along $\varphi = 0$ and $\varphi = \bar{\beta}$.

For the stress function F_1 both differential equation and boundary conditions are the same as for the deflection w_1. Hence, upon putting as in eq.(I-11-19),

$$\begin{aligned} 2F_1^h &= \bar{z}\Phi(z) + \overline{z\Phi(z)} + X(z) + \overline{X(z)} \\ \Psi(z) &= X'(z) \end{aligned} \tag{4-11}$$

one has at once from eqs.(3-4)

$$\varkappa\Phi(z) - \overline{z\Phi'(z)} - \overline{\Psi(z)} = \left(\frac{\partial F_1^p}{\partial r} + i\,\frac{1}{r}\,\frac{\partial F_1^p}{\partial\varphi}\right)e^{i\varphi} \tag{4-12}$$

where

$$\varkappa = \frac{3-\nu}{1+\nu} \tag{4-13}$$

5. Example. Consider the particular case where the temperature increases from its initial uniform value zero to a constant value T_0 in the upper half of the wing and, at the same time, decreases to the value $-T_0$ in the lower half. Then

$$\tau_0 = (1+\nu)\alpha T_0, \qquad \tau_1 = 0$$

and, from eqs.(4-1), taking symmetry with respect to $x = 0$ into account,

$$w_1^p = 0, \quad F_1^p = -\frac{C}{2}(x^2+y^2) = -\frac{C}{2} z\bar{z} \tag{5-1}$$

where

$$C = \frac{1-\nu^2}{2} D\alpha T_0 \tag{5-2}$$

Substitution into eq.(4-10) renders $w_1^h = 0$ and hence $w_1 = 0$, while from eq.(4-12) one has

$$\varkappa\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)} = -Cz \tag{5-3}$$

The solution of this equation can easily be given in closed form:

$$\Phi(z) = -\frac{C}{\varkappa-1} z, \quad \Psi(z) = 0 \tag{5-4}$$

Eq.(5-3) is then satisfied not only on the boundary of the triangular region but everywhere within this region. From eq.(4-11) one finds

$$F_1^h = -\frac{C}{\varkappa-1} z\bar{z} \tag{5-5}$$

and hence, using eq.(5-1),

$$F_1 = - \frac{C}{2} \frac{\varkappa+1}{\varkappa-1} z\bar{z} \tag{5-6}$$

The corresponding state of stress is uniform compression in the upper half and uniform tension in the lower half of the wing. In particular, one finds in the upper shell

$$n_{x1} = \frac{\partial^2 F_1}{\partial y^2} = - \frac{\varkappa+1}{\varkappa-1} C = n_{y1}, \quad n_{xy1} = 0 \tag{5-7}$$

In the case of a more complicated temperature distribution no closed solution can, in general, be found and a numerical procedure has to be used. In order to demonstrate this the problem stated above will now be solved numerically.

Eq.(5-3) is transformed from the triangular region into the circular domain by means of the mapping function $z = \omega(\zeta)$, cf.eq.(I-11-6),

$$\varkappa\Phi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\Phi'(\sigma)} - \overline{\Psi(\sigma)} = CH(\sigma) = - C\omega(\sigma) \tag{5-8}$$

valid on the boundary $\zeta = \sigma$ of the unit circle. Using the same expansions as in eqs.(I-11-8a) and (I-11-22) to (I-11-24) and utilizing the Cauchy formula (I-11-12) the following equation is obtained

$$\varkappa\sum_{1}^{\infty} A_k\zeta^k - \sum_{k=0}^{\infty} \sum_{m=1}^{n+1} m\,\bar{A}_m\, c_{m+k-1}\, \zeta^k - \bar{B}_o = C\sum_{0}^{\infty} H_k\zeta^k$$

Comparison of coefficients of equal powers of ζ^k renders the following set of linear equations for the expansion

coefficients A_k of function $\Phi(\zeta)$

$$c_o\bar{A}_1 + 2c_1\bar{A}_2 + \dots + (n+1)c_n\,\bar{A}_{n+1} + \bar{B}_o = -H_oC \qquad (5\text{-}9)$$

and

$$\varkappa A_k - \sum_{m=1}^{n} m\,c_{m+k-1}\,\bar{A}_m = H_kC \qquad (k = 1,2,..n) \qquad (5\text{-}10)$$

A similar set of equations for the coefficients B_k of the function $\Psi(\zeta)$ is obtained by replacing eq.(5-8) by its conjugate complex and performing the same operations as before. One gets

$$-B_k = C\bar{H}_{-k} + \sum_{m=1}^{n} m\bar{c}_{m-k-1}\,A_m \qquad (k=1,2\dots 2n-1) \qquad (5\text{-}11)$$

Coefficients c_n are all real quantities and have already been tabulated in Part I, Table 12, for a set of 10 triangles. Coefficients H_k of the expansion (I-11-22)

$$H(\sigma) = \sum_{-n}^{+n} H_k\sigma^k \qquad (5\text{-}12)$$

have to be calculated by means of Fourier expansion of $H(\sigma)$ along the unit circle, cf.sec.I-16. In the present example we have

$$H(\sigma) = -\omega(\sigma) = -Re^{i\gamma} = -R(\cos\gamma + i\sin\gamma)$$

Real and imaginary part of $H(\sigma)$ as functions of the polar angle ϑ on the circumference of the unit circle can be taken from tables I-1 to I-10, cf. Fig.I-10. As in the example in Part I a triangle with angle $2\alpha = 100^o$ is assumed.

Figs.3 and 4 show the two functions $-\mathrm{Re}\left\{H(\sigma)\right\}$ and $-\mathrm{Im}\left\{H(\sigma)\right\}$. A Fourier analysis renders the corresponding Fourier coefficients which in turn determine the coefficients H_k, by means of the equations

$$H_o = ia_o, \quad H_n = \tfrac{1}{2}(a_n - b_n), \quad H_{-n} = \tfrac{1}{2}(a_n + b_n) \qquad (5\text{-}13)$$

The Fourier coefficients are given on Figs.3 and 4 for n = 6 while the coefficients H_k are contained in Table 1. The approximation curves are also represented (broken lines).

	k=0	1	2	3	4	5	6	μ
H_k	-0,26i	-10,30i	1,80i	0,43i	-1,94i	0,68i	0,19i	0,047671
H_{-k}		0	-0,08i	0,07i	0,02i	-0,08i	0,07i	

Table 1. Power series coefficients of H
Each value in the tables 1 and 2 is to be multiplied by the corresponding factor μ.

The exact value of all quantities H_{-k} is zero. Table 1 reflects the error involved in the approximation of the mapping function.

Eqs.(5-9), (5-10) and (5-11) can now be set up and solved. The results are given in Table 2. One notices that the A_k are not precisely proportional to the H_k and the B_k are not all zero as would be the case with the exact solution. The error is due to the relatively small number n = 6 of terms in the series expansion.

	k=0	1	2	3	4	5	6	μ
A_k		-8,874i	1,417i	0,308i	-1,334i	0,459i	0,066i	
B_k	-0,770i	-0,677i	2,086i	-0,117i	-1,369i	4,818i	-0,530i	0,047670i
	k=7	8	9	10	11			
B_k	-0,579i	0,356i	-0,076i	-0,107i	-0,012i			

Table 2. Power series coefficients of Φ and Ψ.

Second Approximation

6.) Symmetric temperature distribution. As can be seen from a comparison of eqs.(I-9-4) and the corresponding boundary conditions (I-9-6) with eqs.(I-9-3) and boundary conditions (I-9-5) the steps to be performed in the second approximation are, in principle, the same as in the first approximation. They deviate, however, in two respects. First, the equations for the particular solutions w_2^p and F_2^p can no longer be reduced to the simple form (I-10-2). Second, a line load has to be placed along the axis of y in order to have the boundary conditions satisfied there.

The determination of the particular solutions

$$\left.\begin{aligned} N\nabla^2\nabla^2 w_2^p &= -\frac{1}{r}\left(\frac{\partial^2 F_1^p}{\partial r^2} + \frac{\partial^2 F_1^h}{\partial r^2}\right) \\ \nabla^2\nabla^2 F_2^p &= \frac{Eh}{r}\left(\frac{\partial^2 w_1^p}{\partial r^2} + \frac{\partial^2 w_1^h}{\partial r^2}\right) \end{aligned}\right\} \qquad (6\text{-}1)$$

may be performed either in the triangular z-plane or in the

circular ζ-plane, where $z = \omega(\zeta)$. Since w_1^p and F_1^p are usually known in the z-plane a series expansion similar to (I-10-5) and (I-10-7) may be used for the corresponding parts w_{21}^p and F_{21}^p of w_2^p and F_2^p:

$$\left.\begin{aligned} w_{21}^p &= \bar{Q}_{mn}\, \bar{r}^{m+k} \cos \frac{n\pi\bar{\varphi}}{\alpha} \\ F_{21}^p &= \bar{P}_{mn}\, \bar{r}^{m+k} \cos \frac{n\pi\bar{\varphi}}{\alpha} \end{aligned}\right\} \qquad (6\text{-}2)$$

and, in case where $m + k = \frac{n\pi}{\alpha}$,

$$\left.\begin{aligned} w_{21}^p &= \bar{Q}_m \bar{r}^{m+k} \ln \frac{\bar{r}}{1} \cos(m+k)\bar{\varphi} \\ F_{21}^p &= \bar{P}_m \bar{r}^{m+k} \ln \frac{\bar{r}}{1} \cos(m+k)\bar{\varphi} \end{aligned}\right\} \qquad (6\text{-}3)$$

In contradistinction, w_1^h and F_1^h are known in the ζ-plane. The corresponding parts w_{22}^p and F_{22}^p of w_2^p and F_2^p may therefore more conveniently be represented by series expansions in terms of ϱ and ϑ . Using the relations

$$\left.\begin{aligned} \nabla^2 &= 4\,\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{4}{\omega'(\zeta)\overline{\omega'(\zeta)}}\,\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}} = \frac{1}{\omega'(\zeta)\overline{\omega'(\zeta)}}\,\Delta = \Theta(\varrho,\vartheta)\Delta \\ \Delta &= \frac{\partial^2}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial\vartheta^2}\,, \quad \Theta(\varrho,\vartheta) = \frac{1}{\omega'(\zeta)\overline{\omega'(\zeta)}} \end{aligned}\right\} \qquad (6\text{-}4)$$

and taking eq.(4-8) into account eqs.(6-1) may be written

$$N\Delta(\Theta\Delta w_{22}^p) = \frac{-1}{r\Theta}\,\mathrm{Re}\left\{\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\,\Phi^{*\prime}(\zeta) + \Psi^{*}(\zeta)\right] e^{2i\varphi} + 2\Phi^{*}(\zeta)\right\} \qquad (6\text{-}5)$$

$$\Delta(\theta\Delta F^{p}_{22}) = \frac{Eh}{r\theta}\,\mathrm{Re}\left\{\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\,\mu^{*\prime}(\zeta) + \gamma^{*}(\zeta)\right]e^{2i\varphi} \dotplus 2\mu^{*}(\zeta)\right\} \qquad (6\text{-}6)$$

where, in accordance with the notation of eqs.(I-11-28),

$$\Phi^{*}(\zeta) = \frac{\Phi'(\zeta)}{\omega'(\zeta)},\Psi^{*}(\zeta) = \frac{\Psi'(\zeta)}{\omega'(\zeta)}, \quad \mu^{*}(\zeta) = \frac{\mu'(\zeta)}{\omega'(\zeta)}, \quad \psi^{*}(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)}$$

In order to evaluate the right-hand side of eqs.(6-5) and (6-6) the function $1/\omega'(\zeta)$ has to be expanded into a series

$$\frac{1}{\omega'(\zeta)} = \sum \alpha_n \zeta^n = \sum \alpha_n \rho^n e^{in\vartheta} \qquad (6\text{-}7)$$

The coefficients α_n in this expansion may be found by putting $\rho = 1$ and using eq.(I-13-2) which gives the values of ω' along the circumference of the unit circle. From eq.(6-4) one has then

$$\theta(\rho,\vartheta) = \sum\sum \alpha_n \bar{\alpha}_m \rho^{n+m} e^{i(n-m)\vartheta} = \sum_r \sum_s \beta_{rs}\, \rho^r \cos s\vartheta \qquad (6\text{-}8)$$

The solution of eqs.(6-5) and (6-6) proceeds now in the following manner. The numerical values of the right-hand sides of these equations are calculated at a sufficient number of points in the ζ-plane using a polar coordinate network ρ,ϑ. From these values a series expansion is obtained with the aid of some suitable numerical method:

$$\left.\begin{aligned} &\frac{-1}{r\theta}\,\mathrm{Re}\left\{\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\,\Phi^{*\prime}(\zeta) + \Psi^{*}(\zeta)\right]e^{2i\varphi} + 2\Phi^{*}(\zeta)\right\} = N\sum\sum A_{mn}\rho^{m-2}\cos n\vartheta \\ &\frac{Eh}{r\theta}\,\mathrm{Re}\left\{\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\,\mu^{*\prime}(\zeta) + \gamma^{*}(\zeta)\right]e^{2i\varphi} + 2\mu^{*}(\zeta)\right\} = \sum\sum a_{mn}\rho^{m-2}\cos n\vartheta \end{aligned}\right\} \qquad (6\text{-}9)$$

The expansion will contain cosine terms only since all functions are symmetric with respect to the y-axis. A similar expansion - with unknown coefficients - is then assumed for the following quantities

$$\left.\begin{aligned} \theta\Delta w_{22}^{p} &= \sum_{m}\sum_{n}{}' b_{mn}\rho^{m}\cos n\vartheta + \sum_{n} b_{n}\rho^{n}\ln\rho\cos n\vartheta \\ \theta\Delta F_{22}^{p} &= \sum_{m}\sum_{n}{}' B_{mn}\rho^{m}\cos n\vartheta + \sum_{n} B_{n}\rho^{n}\ln\rho\cos n\vartheta \end{aligned}\right\} \quad (6\text{-}10)$$

where the prime at the summation sign indicates that terms with $m = n$ have to be omitted. From the definition of the operator Δ, eq.(6-4), there follows

$$\Delta(\theta\Delta w_{22}^{p}) = \sum_{m}\sum_{n}{}'(m^2-n^2)b_{mn}\rho^{m-2}\cos n\vartheta + 2\sum_{n} n\, b_{n}\rho^{n-2}\cos n\vartheta$$

$$\Delta(\theta\Delta F_{22}^{p}) = \sum_{m}\sum_{n}{}'(m^2-n^2)B_{mn}\rho^{m-2}\cos n\vartheta + 2\sum_{n} n\, B_{n}\rho^{n-2}\cos n\vartheta$$

Comparing coefficients with eq.(6-9) one finds

$$\left.\begin{aligned} b_{mn} &= \frac{A_{mn}}{m^2-n^2} \\ B_{mn} &= \frac{a_{mn}}{m^2-n^2} \end{aligned}\right\} (m \neq n) \qquad \left.\begin{aligned} b_{n} &= \frac{A_{nn}}{2n} \\ B_{n} &= \frac{a_{nn}}{2n} \end{aligned}\right\} \quad (6\text{-}11)$$

In order to obtain w_{22}^{p} and F_{22}^{p} the foregoing steps would have to be repeated. This, however, leads to very time-consuming calculations. We proceed therefore in an approximate fashion by applying the same method used to obtain eq. (6-9), i.e. we evaluate eqs.(6-10) and (6-8) at all points of the coordinate

network and expand:

$$\left.\begin{aligned} \Delta w_{22}^{p} &= \sum\sum c_{mn}\rho^{m-2} \cos n\vartheta \\ \Delta F_{22}^{p} &= \sum\sum C_{mn}\rho^{m-2} \cos n\vartheta \end{aligned}\right\} \quad (6\text{-}12)$$

Then, upon putting

$$\left.\begin{aligned} w_{22}^{p} &= \sum\sum{}' w_{mn}\rho^{m} \cos n\vartheta + \sum w_{n}\rho^{n}\ln\rho \cos n\vartheta \\ F_{22}^{p} &= \sum\sum{}' F_{mn}\rho^{m} \cos n\vartheta + \sum F_{n}\rho^{n}\ln\rho \cos n\vartheta \end{aligned}\right\} (6\text{-}13)$$

and substituting into eqs.(6-12) one finds, corresponding to eqs.(6-11),

$$\left.\begin{aligned} w_{mn} &= \frac{c_{mn}}{m^2-n^2} \\ F_{mn} &= \frac{C_{mn}}{m^2-n^2} \end{aligned}\right\} (m \neq n) \qquad \left.\begin{aligned} w_{n} &= \frac{c_{nn}}{2n} \\ F_{n} &= \frac{C_{nn}}{2n} \end{aligned}\right\} \quad (6\text{-}14)$$

To w_2^p a second solution pertaining to the line load (3-7) has to be added. The solution for an infinite plate strip simply supported along its edges and carrying a line load is available in the literature[1] and may be used here. Introducing a coordinate system x,y as shown in Fig.1 one has

$$w_2^q = \frac{l^3}{4N\pi^3}\sum_n \frac{a_n}{n^3}\left(1 + \frac{n\pi|x|}{l}\right)e^{-\frac{n\pi|x|}{l}} \sin\frac{n\pi y^*}{l} \quad (6\text{-}15)$$

[1] Girkmann, p.170.

where $y^* = y + l_o$. The coefficients a_n are those of the Fourier expansion of the line load,

$$q(y^*) = -2l \frac{\partial f}{\partial x} n_{x1} \Big|_{x=0} = \sum a_n \sin \frac{n\pi y^*}{l} \qquad (6\text{-}16)$$

For a supersonic wing profile the following approximation may be used

$$lf = (\bar\beta^s - \varphi^s)y^* = (\bar\beta^s - \varphi^s) r \sin\varphi \qquad (6\text{-}17)$$

where s is a suitable chosen positive number. Then

$$l\frac{\partial f}{\partial x} = s\varphi^{s-1}\sin^2\varphi, \quad l\frac{\partial f}{\partial y} = \bar\beta^s - \varphi^{s-1}(\varphi + \frac{s}{2}\sin 2\varphi) \qquad (6\text{-}18)$$

The ratio of the leading-edge slope to the trailing-edge slope, taken along sections x = konst, is given by $s \sin 2\bar\beta / 2\bar\beta$.

The normal force n_{x1} at $x = 0$ is the sum of the particular part n_{x1}^p and of the homogeneous part n_{x1}^h. The first is in terms of F_1^p, eq.(I-10-5), given by

$$n_{x1}^p = \frac{\partial^2 F_1^p}{\partial y^2} = \frac{\partial^2 F_1^p}{\partial \bar r^2}\Big|_{\bar\varphi = 0} \qquad (6\text{-}19)$$

while for the second one finds from eqs.(I-11-30)

$$n_{x1}^h = n_{\vartheta 1}^h\Big|_{\vartheta=0} = \Phi_1^*(\zeta) + \overline{\Phi_1^*(\zeta)} + \frac{1}{\overline{\omega'(\zeta)}}\left[\overline{\omega(\zeta)}\Phi_1^{*\prime}(\zeta) + \Psi_1'(\zeta)\right] \qquad (6\text{-}20)$$

The coefficients a_n in the expansion (6-16) may now be calculated by some suitable numerical method.

The homogeneous solutions w_2^h and F_2^h of the second approximation

are introduced in the same form as in the first approximations, eqs.(I-11-3) and (I-11-19)

$$\left.\begin{aligned} 2w_2^h &= \bar{z}\mu(z) + z\overline{\mu(z)} + \chi(z) + \overline{\chi(z)} \\ 2F_2^h &= \bar{z}\Phi(z) + z\overline{\Phi(z)} + X(z) + \overline{X(z)} \end{aligned}\right\} \qquad (6\text{-}21)$$

The corresponding boundary conditions, eqs(I-9-6), are identical with those of the first approximation, eqs(I-9-5). Hence, eq.(I-11-4) remains unchanged:

$$\mu(z) + z\overline{\mu'(z)} + \overline{\psi(z)} = -\left(\frac{\partial w_2}{\partial x} + i\,\frac{\partial w_2}{\partial y}\right)^{p+q} \qquad (6\text{-}22)$$

or, after transformation into the ζ-plane,

$$\mu(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)}\,\overline{\mu'(\sigma)} + \overline{\psi(\sigma)} = -\left(\frac{\partial w_2}{\partial x} + i\,\frac{\partial w_2}{\partial y}\right)^{p+q} = G(\sigma) \qquad (6\text{-}23)$$

Similarly, for F_2^h

$$\Phi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)}\,\overline{\Phi'(\sigma)} + \overline{\Psi(\sigma)} = -\left(\frac{\partial F_2}{\partial x} + i\,\frac{\partial F_2}{\partial y}\right)^{p} = H(\sigma) \qquad (6\text{-}24)$$

We note that

$$\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y} = e^{i\varphi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\,\frac{\partial}{\partial \varphi}\right) \qquad (6\text{-}25)$$

<u>7. Skew-symmetric temperature distribution.</u> Eqs.(3-2) are valid in this case. They are identical with those of the symmetric distribution. Moreover, the same line load as

in the symmetric case given by eq.(6-16) has to be placed on the axis of y. Boundary conditions, however, are now in the form of eqs.(3-5) and (3-6).

The particular integrals $w_2^p + w_2^q$ and F_2^p are determined in exactly the same way as in the symmetric case, sec.6.

For the homogeneous solution w_2^h one has from boundary conditions (3-5) on $\varphi = 0$ and $\varphi = \bar{\beta}$, with $w_2^p + w_2^q = w_2^*$,

$$\left.\begin{aligned} &\nabla^2 w_2^h - (1-\nu)\frac{\partial^2 w_2^h}{\partial r^2} = (1-\nu)\frac{\partial^2 w_2^*}{\partial r^2} - \nabla^2 w_2^* \\ &\frac{\partial}{\partial\varphi}\left[\frac{1}{r}\nabla^2 w_2^h + (1-\nu)\frac{\partial^2}{\partial r^2}\left(\frac{w_2^h}{r}\right)\right] \\ &= \frac{1}{N}\left(\frac{1}{r}\frac{\partial f}{\partial\varphi}\right)\frac{\partial^2 F_1}{\partial r^2} - \frac{\partial}{\partial\varphi}\left[\frac{1}{r}\nabla^2 w_2^* + (1-\nu)\frac{\partial^2}{\partial r^2}\left(\frac{w_2^*}{r}\right)\right] \end{aligned}\right\} \qquad (7\text{-}1)$$

w_2^h is now split up into two parts,

$$w_2^h = w_{21}^h + w_{22}^h \qquad (7\text{-}2)$$

where

$$2w_{21}^h = \bar{z}\mu_1(z) + z\overline{\mu_1(z)} + \chi_1(z) + \overline{\chi_1(z)} \qquad (7\text{-}3)$$

$$2w_{22}^h = \bar{z}\mu_2(z) + z\overline{\mu_2(z)} + \chi_2(z) + \overline{\chi_2(z)} \qquad (7\text{-}4)$$

The first part w_{21}^h is obtained as a biharmonic function satisfying boundary conditions (4-2) or (4-10), with w_1^p replaced by w_2^*. The second part w_{22}^h is then subject to the

following boundary conditions

$$\left.\begin{aligned} &\nabla^2 w_{22}^h - (1-\nu)\frac{\partial^2 w_{22}^h}{\partial r^2} = -\nabla^2 w_2^* \\ &\frac{\partial}{\partial\varphi}\left[\frac{1}{r}\nabla^2 w_{22}^h + (1-\nu)\frac{\partial^2}{\partial r^2}\left(\frac{w_{22}^h}{r}\right)\right] = \frac{1}{N}\left(\frac{1}{r}\frac{\partial f}{\partial\varphi}\right)\frac{\partial^2 F_1}{\partial r^2} - \frac{\partial}{\partial\varphi}\left(\frac{1}{r}\nabla^2 w_2^*\right) \end{aligned}\right\} \quad (7\text{-}5)$$

We proceed now as in sec.4. The second of eqs.(7-5) is multiplied by dr and integrated. Bearing in mind that $\partial f/r\partial\varphi$ is independent of r, cf. eq.(6-17), and putting the integration constant equal to zero, we obtain

$$2i\left[\mu_2'(z) - \overline{\mu_2'(z)}\right] + \frac{i}{2}(1-\nu)\left\{\left[\bar{z}\mu_2''(z) + \psi_2'(z)\right]e^{2i\varphi} - \left[z\overline{\mu_2''(z)} + \overline{\psi_2'(z)}\right]e^{-2i\varphi}\right\} = \frac{1}{N}\left(\frac{1}{r}\frac{\partial f}{\partial\varphi}\right)\frac{\partial F_1}{\partial r} - \int\frac{1}{r}\frac{\partial}{\partial\varphi}(\nabla^2 w_2^*)dr$$

This equation is multiplied by i and then subtracted from the first of eqs.(7-5), to yield

$$\begin{aligned} (3+\nu)\mu_2'(z) - (1-\nu)\overline{\mu_2'(z)} - (1-\nu)\left[z\overline{\mu_2''(z)} + \overline{\psi_2'(z)}\right]e^{-2i\varphi} &= \\ &= -\int\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\varphi}\right)\nabla^2 w_2^* dr - \frac{i}{N}\left(\frac{1}{r}\frac{\partial f}{\partial\varphi}\right)\frac{\partial F_1}{\partial r} \\ &= -2\int e^{i\varphi}\frac{\partial}{\partial z}(\nabla^2 w_2^*)dr - \frac{i}{N}\left(\frac{1}{r}\frac{\partial f}{\partial\varphi}\right)\frac{\partial F_1}{\partial r} \\ &= -2\nabla^2 w_2^* - \frac{i}{N}\left(\frac{1}{r}\frac{\partial f}{\partial\varphi}\right)\frac{\partial F_1}{\partial r} \end{aligned}$$

Another integration with respect to r renders

$$\lambda \mu_2(z) - z\overline{\mu'(z)} - \overline{\psi_2(z)} = -\left[\frac{i}{N}\left(\frac{1}{r}\frac{\partial f}{\partial \varphi}\right)F_1 + 2\int \nabla^2 w_2^* dr\right]\frac{e^{i\varphi}}{1-\nu} \quad (7\text{-}6)$$

where λ is given by eq.(4-9).

The same procedure may be applied to boundary condition (3-6). Letting

$$F_2^h = F_{21}^h + F_{22}^h \quad (7\text{-}7)$$

and putting

$$2F_{21}^h = \bar{z}\Phi_1(z) + z\overline{\Phi_1(z)} + X_1(z) + \overline{X_1(z)} \quad (7\text{-}8)$$

$$2F_{22}^h = \bar{z}\Phi_2(z) + z\overline{\Phi_2(z)} + X_2(z) + \overline{X_2(z)} \quad (7\text{-}9)$$

one has boundary condition (4-12), with F_1^p replaced by F_2^p, valid for F_{21}^h while F_{22}^h has to satisfy the following equation

$$\varkappa\Phi_2(z) - z\overline{\Phi_2'(z)} - \overline{\Psi_2(z)} = -2\left[\int \nabla^2 F_2^p dr\right]\frac{e^{i\varphi}}{1+\nu} \quad (7\text{-}10)$$

where $\varkappa$ is given by eq.(4-13).

8. Example. The example of sec.5 shall now be continued into the second approximation. With $w_1 = 0$ one finds at once from the second of eqs.(3-2) together with boundary conditions (3-6)

$$F_2 = 0 \quad (8\text{-}1)$$

In order to determine w_2 one has, however, to perform all the steps outlined in sec.7. First the particular solution w_2^p must be found. Substituting eq.(5-6) into the first of eqs. (6-1) one readily gets

$$w_2^p = \frac{C}{9N} \frac{\varkappa+1}{\varkappa-1} r^3 + B(r) \qquad (8-2)$$

where B(r) is some arbitrary biharmonic function which will be determined later.

Next, the solution w_2^q due to the line load (3-7) has to be calculated. Substituting n_{x1} from eq.(5-7) into eq.(6-16) and using the first of eqs.(6-18) one obtains

$$q(y^*) = 2C \frac{\varkappa+1}{\varkappa-1} s\varphi^{s-1}\sin^2\varphi \qquad (8-3)$$

A value of s = 0.08 has been chosen in the following. Fig.5 shows the corresponding cross section of the wing along x = const. Two other types with s = 0.07 and 0.1, respectively, are also shown. The relative maximum thickness of the wing in the three cases is 4.0 %, 4.6 % and 5.6 %, respectively.

In accordance with eq.(6-16) expression (8-3) is now developed into a Fourier sine series. As has been done consequently in this example only the first 6 terms in the series are retained. Table 3 shows the corresponding coefficients a_k, with $\nu = 0.3$ in the expression $(\varkappa+1)/(\varkappa-1) \equiv 2/(1-\nu)$.

k	1	2	3	4	5	6
a_k/C	0.1959	-0.0868	0.0556	-0.0426	0.0333	-0.0280

Table 3.

For the calculation of the first part w_{21}^h of the homogeneous solution w_2^h eq.(4-10) is used in the form

$$\lambda\mu_1(z) - z\overline{\mu_1'(z)} - \overline{\psi_1(z)} = \frac{\partial w_2^*}{\partial x} + i\frac{\partial w_2^*}{\partial y} = G^{(1)}(\sigma) \qquad (8-4)$$

ϑ°	$\partial w_2^p/\partial x$	$\partial w_2^p/\partial y$	$\partial w_2^q/\partial x$	$\partial w_2^q/\partial y$	$\partial w_2^*/\partial x$	$\partial w_2^*/\partial y$
0	181.87	152.62	0	-2.55	181.87	150.07
7.5	95.19	79.69	0.18	-2.29	95.37	77.40
15	59.49	49.72	0.31	-2.06	59.80	47.66
30	28.48	23.82	0.48	-1.69	28.96	22.13
45	4.65	3.90	0.58	-1.38	5.23	2.52
60	-15.83	-13.27	0.66	-1.08	-15.17	-14.35
75	-35.29	-29.54	0.71	-0.78	-34.58	-30.32
90	-52.94	-44.41	0.72	-0.46	-52.22	-44.87
105	-68.51	-57.66	0.67	-0.11	-67.84	-57.77
112.5	-89.92	-75.28	0.58	0.12	-89.34	-75.16
120	0	0	0	0.24	0	0.24
127.5	-116.44	0	0	1.15	-116.44	1.15
135	-88.68	0	0	1.43	- 88.68	1.43
150	-64.91	0	0	1.75	- 64.91	1.75
165	-34.77	0	0	1.93	- 34.77	1.92
180	0	0	0	1.99	0	1.99

Table 4.

All values are to be multiplied by C/N.

with $w_2^* = w_2^p + w_2^q$.

Table 4 contains the values of the derivatives of Nw_2^p/C from eq.(8-2), of Nw_2^q/C from eq.(6-15) and of Nw_2^*/C as functions of the polar angle ϑ in the ζ-plane, see Fig.I-6. The as yet undetermined function B(r) in eq.(8-2) has now been chosen in such a way as to make the discontinuity produced by r^3 in $\vartheta = 180^\circ$ vanish. One finds $B(r) = -\frac{C}{6N}\frac{\varkappa+1}{\varkappa-1} br^2$. The corresponding graph is represented in full lines in Figs.6 and 7.

Real and imaginary part of $G^{(1)}(\sigma)$ in eq.(8-4) are now developed into Fourier series, represented by the broken lines in Figs.6 and 7. The coefficients G_n in the expansion. (I-11-9) follow then from the equations

$$G_0^{(1)} = ia_0, \quad G_n^{(1)} = \frac{i}{2}(a_n - b_n), \quad G_{-n}^{(1)} = \frac{i}{2}(a_n + b_n) \qquad (8\text{-}5)$$

They are all purely imaginary and are given in Table 5.

k	-6	-5	-4	-3	-2	-1
$G_k^{(1)}$	6.48	17.28	5.04	8.58	49.14	-15.90

k	0	1	2	3	4	5	6
$G_k^{(1)}$	-0.42	-38.82	4.86	1.02	5.76	1.20	1.92

Table 5.

All values are to be multiplied by $i\,\frac{C}{N}$

Eqs.(8-4) are now solved by introducing expansions for μ_1 and ψ_1 of the form (I-11-10) and (I-11-11). The results are presented in Table 6.

n	0	1	2	3	4	5	6
$a_n^{(1)}$	0	- 9.97	1.24	0.27	1.03	0.32	0.39
$b_n^{(1)}$	-1.86	-17.96	59.60	11.30	8.40	16.88	5.73

n	7	8	9	10	11
$b_n^{(1)}$	-1.98	- 0.82	-0.36	-0.28	-0.07

Table 6.

All values are to be multiplied by $i\,\frac{C}{N}$

For the calculation of the second part w_{22}^h of the homogeneous solution the foregoing procedure has to be applied to eq.(7-6). Using eq.(6-15) one finds

$$\nabla^2 w_2^q = -\frac{1}{2N\pi} \sum \frac{a_n}{n}\, e^{-\frac{n\pi|x|}{l}} \sin \frac{n\pi y^*}{l}$$

This expression is zero along the base $y = -l_o$ of the triangle while along the legs one has

$$\int_0^r \nabla^2 w_2^q dr = \frac{l^2 \cos\alpha}{2N\pi^2} \sum \frac{a_n}{n^2} \left[e^{-\frac{n\pi|x|}{l}} \left(\cos \frac{n\pi y^*}{l} - \tan\alpha \sin \frac{n\pi y^*}{l}\right) - e^{-n\pi \tan\alpha} \right]$$

Eq.(8-2) renders

$$\nabla^2 w_2^p = \frac{C}{N}\,\frac{\varkappa+1}{\varkappa-1}\left(r - \frac{2}{3}\,b\right)$$

whence

$$\int_0^r \nabla^2 w_2^p dr = \frac{C}{N}\,\frac{\varkappa+1}{\varkappa-1}\,r\left(\frac{r}{2} - \frac{2}{3}\,b\right)$$

In addition, from eq.(6-17),

$$\frac{1}{r}\,\frac{\partial f}{\partial \varphi} = (\beta^s - \varphi^s)\cos\varphi - s\varphi^{s-1}\sin\varphi$$

Using eq.(5-6) for F_1 the right-hand side of eq.(7-6), denoted by $G^{(2)}(\sigma)$, may now be calculated. Figs.8 and 9 show the graphs of its real and imaginary parts together with the corresponding Fourier expansions (broken lines). From these coefficients $G_k^{(2)}$ are found with the aid of eqs,(8-5) and are represented in Table 7.

k	-6	-5	-4	-3	-2	-1
$G_k^{(2)}$	-9.0	62.1	-70.2	173.0	-200.1	536.0

k	0	1	2	3	4	5	6
$G_k^{(2)}$	457.6	-454.3	-23.7	-159.2	65.4	-125.1	17.4

Table 7

All values are to be multiplied by iC/N.

The last step consists in the determination of the expansion coefficients of the functions $\mu_2(\zeta)$ and $\psi_2(\zeta)$. Table 8 contains the coefficients.

n	0	1	2	3	4	5	6
$a_n^{(2)}$	0	-114.5	- 4.8	-33.2	11.7	-25.9	3.5
$b_n^{(2)}$	408.5	409.3	-115.6	69.6	-15.2	23.2	78.3

n	7	8	9	10	11
$b_n^{(2)}$	14.3	- 0.4	10.4	1.8	- 0.6

Table 8.

All values are to be multiplied by iC/N

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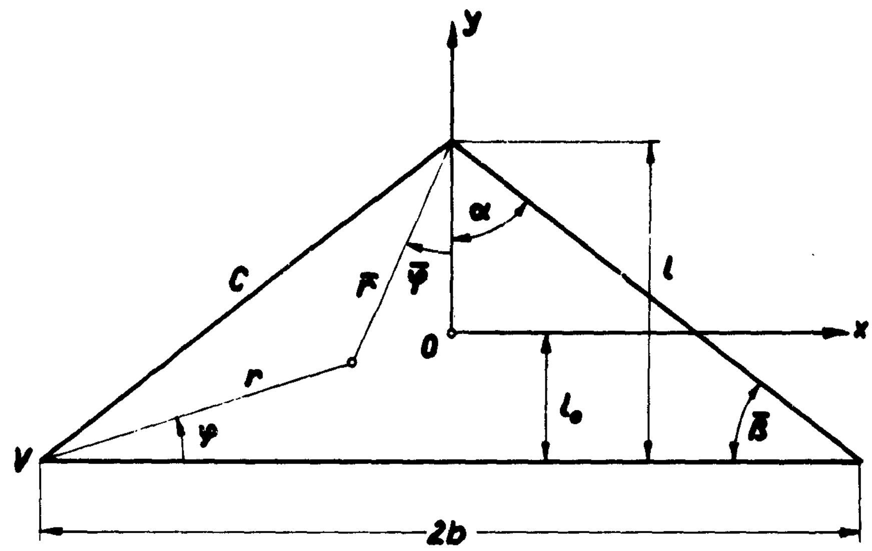


Fig. 1

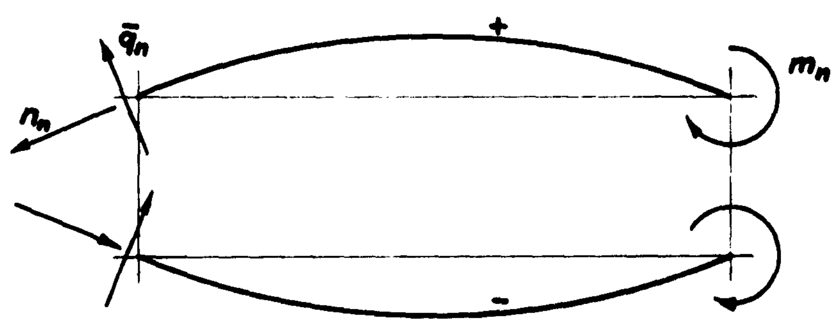
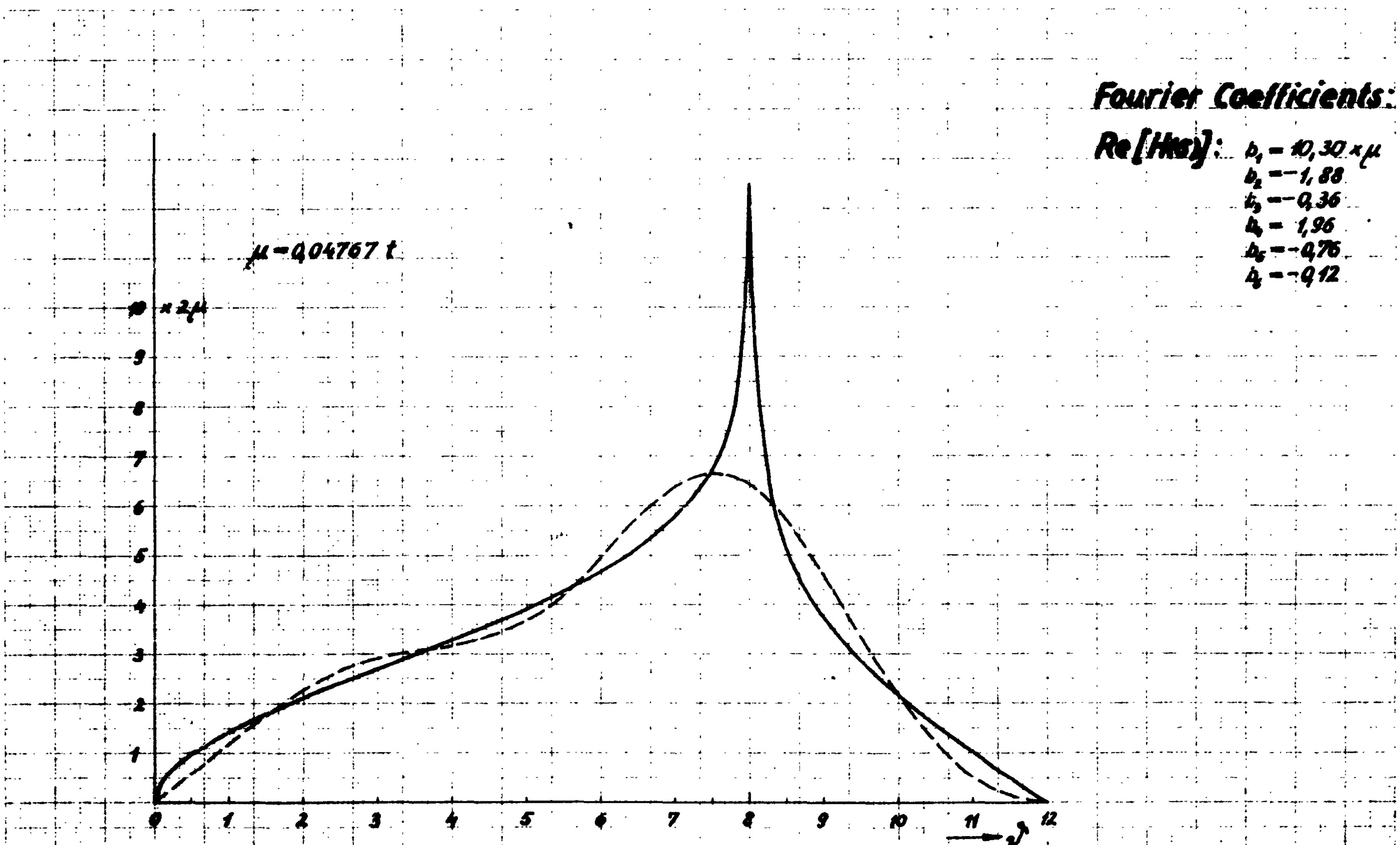
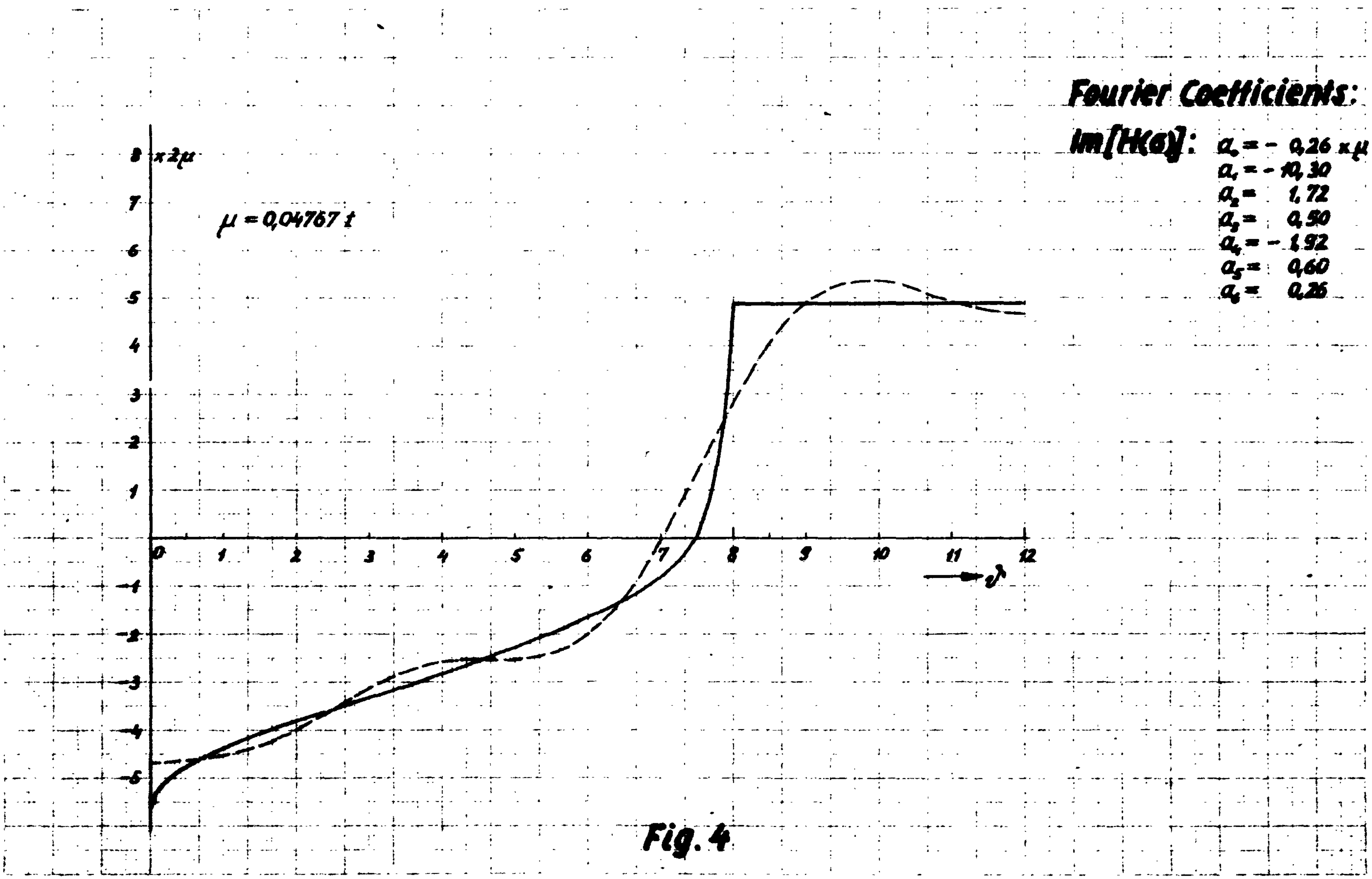


Fig. 2



Fourier Coefficients:
Re[Hsy]: $b_1 = 10,30 \times \mu$
$b_2 = -1,88$
$b_3 = -0,36$
$b_4 = 1,96$
$b_5 = -0,76$
$b_6 = -0,12$
$\mu = 0,04767\ t$
$\times 2\mu$

Fig. 3


Fourier Coefficients:
Im[H(σ)]: a₀ = - 0,26 x μ
a₁ = - 10,30
a₂ = 1,72
a₃ = 0,50
a₄ = - 1,92
a₅ = 0,60
a₆ = 0,26
μ = 0,04767 t
x 2μ
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
0
1
2
3
4
5
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12
ϑ

Fig. 4

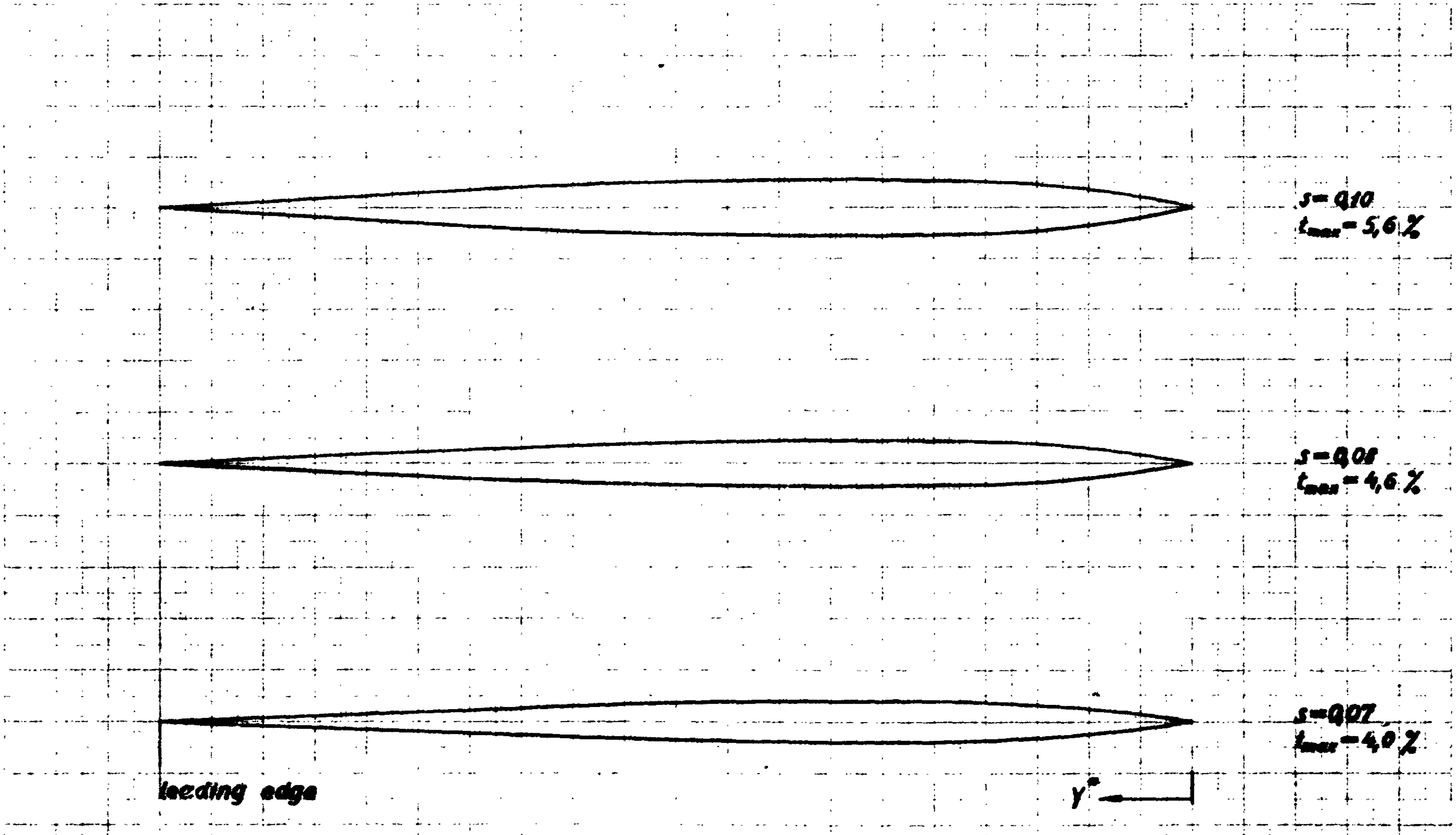
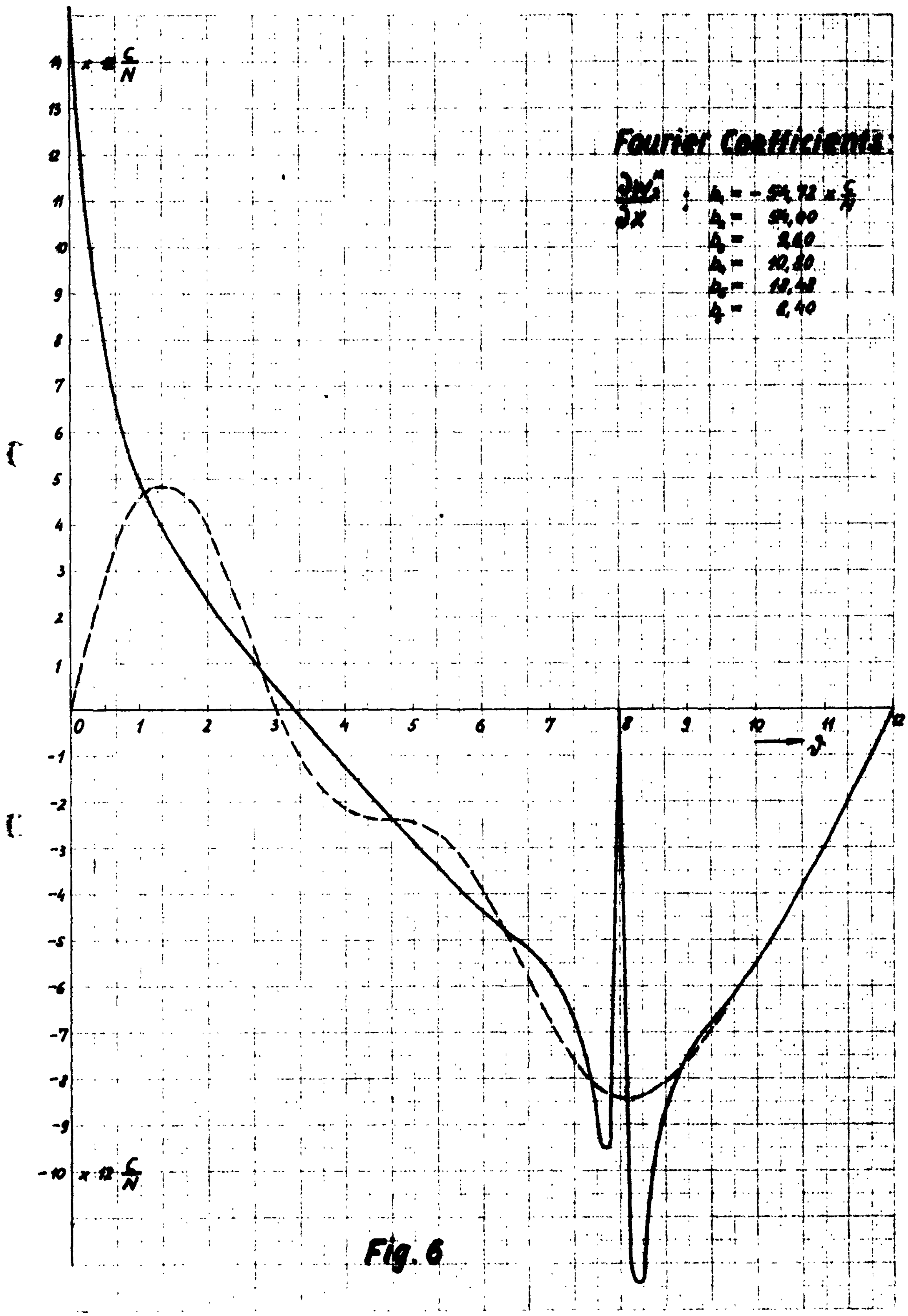


Fig. 5



Fourier Coefficients:
14
13
12
11
10
9
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
0
1
2
3
4
5
6
7
8
9
10
11
12

Fig. 6

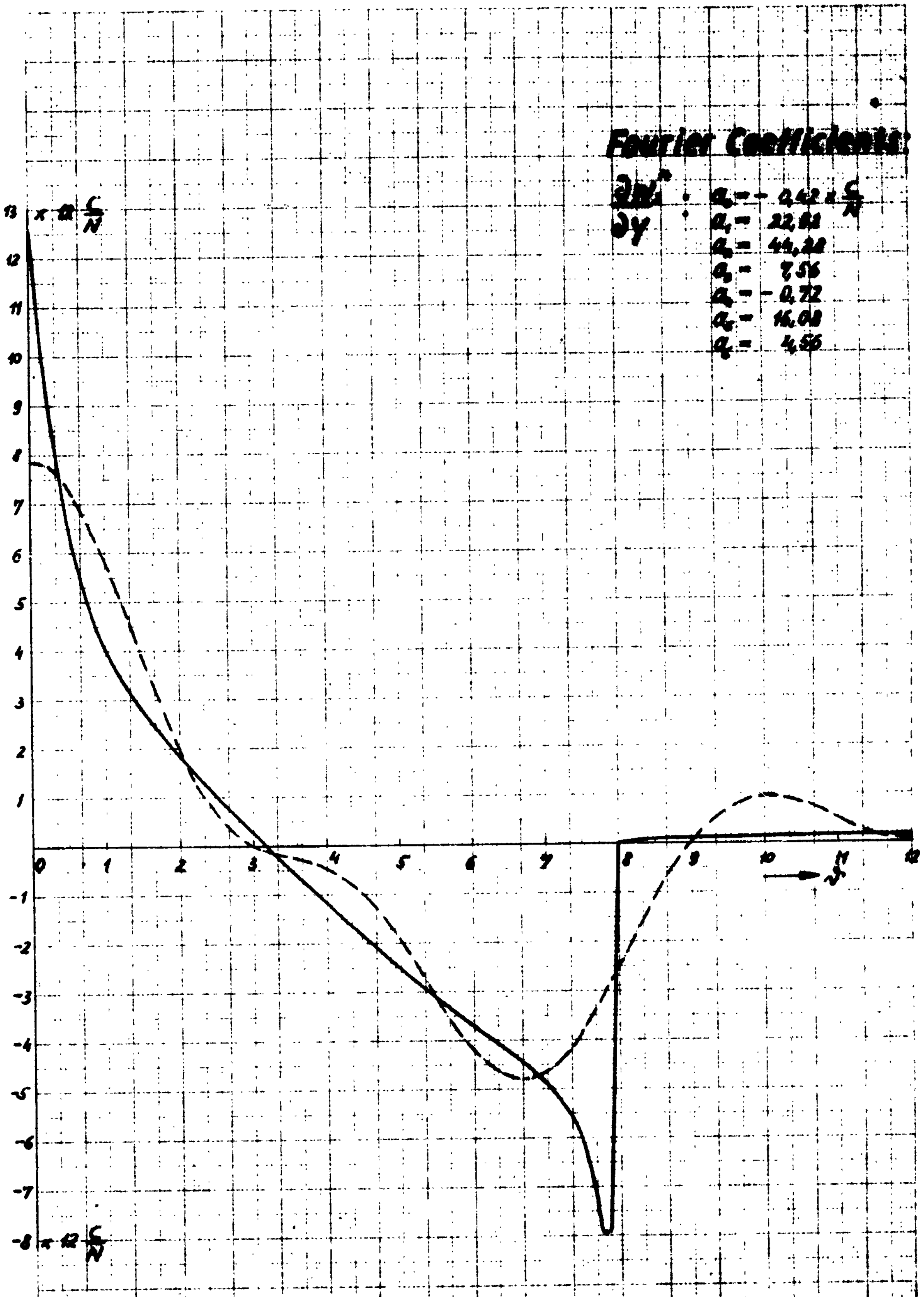

Fourier Coefficients:
$a_0 = -0,42 \times \frac{C}{N}$
$a_1 = 22,82$
$a_2 = 44,28$
$a_3 = 7,56$
$a_4 = -0,72$
$a_5 = 16,08$
$a_6 = 4,56$

Fig.7

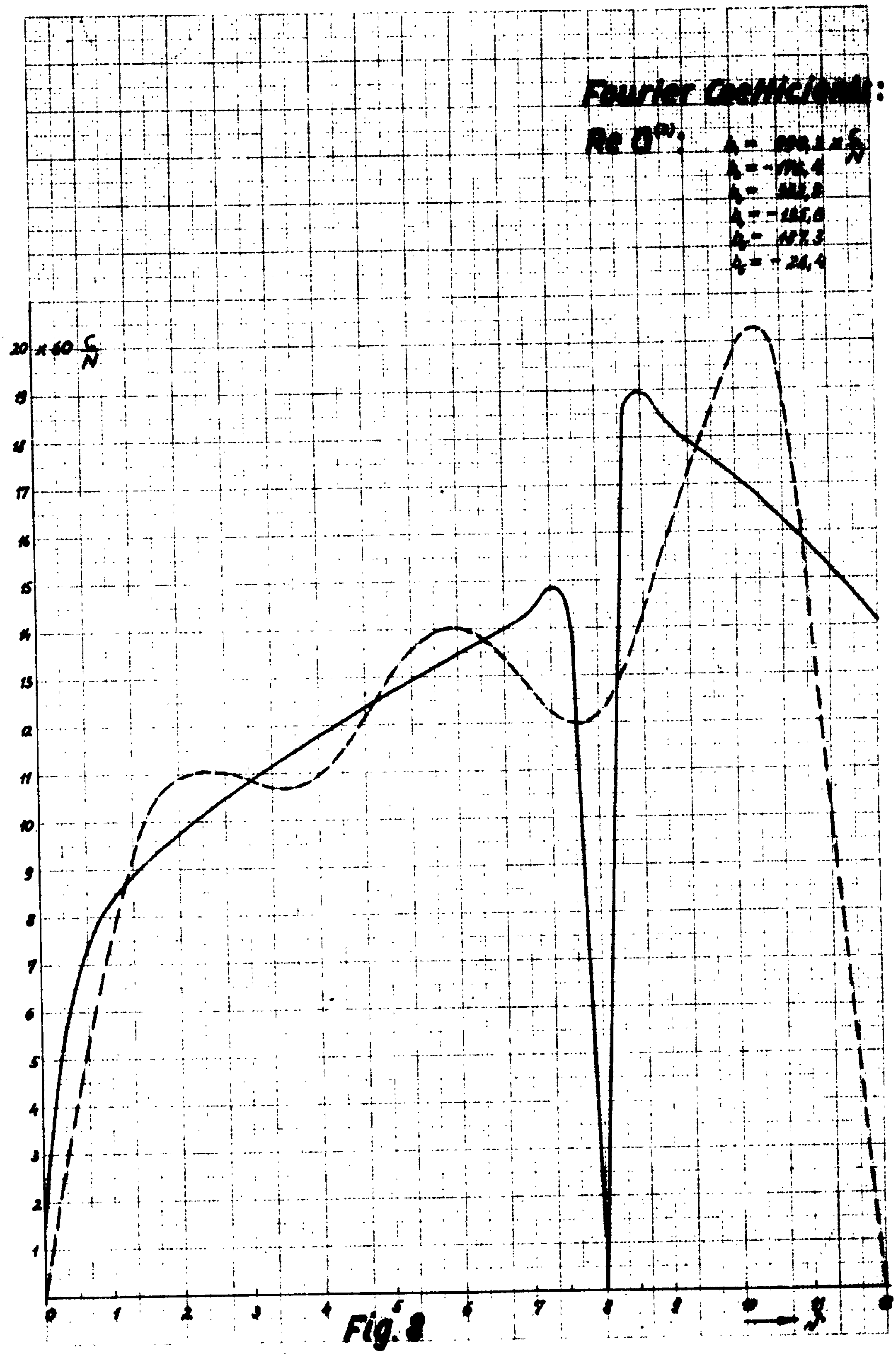

Fourier Coefficients:
20 × 10 C/N
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
11
12

Fig. 8

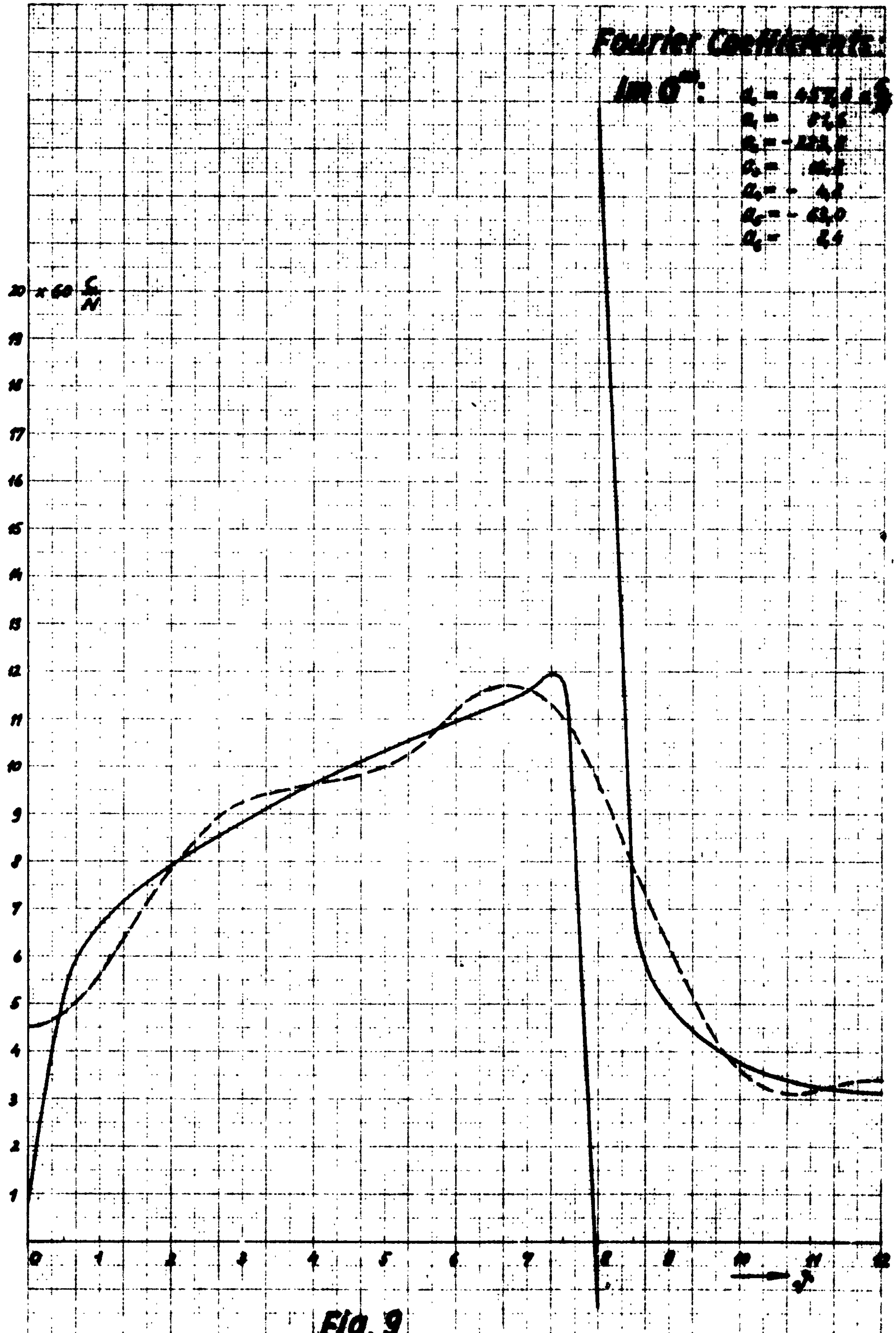


Fig. 9